Model Answer (AS-633)

Pre Ph.D. Course work Examination, 2013-14

Mathematics

Geometry of Finsler spaces

1. (i). Let $F(x^i, y^i)$ be a function of line element (x^i, y^i) which satisfies

- $F(x^i, y^i)$ is positively homogeneous of degree one in y^i.
- $F(x^i, y^i)$ is positive, if not all y^i vanish simultaneously.
- The quadratic form $\dot{\partial}_i \dot{\partial}_j F^2 . \xi^i \xi^j$ is positive for all variables ξ^i.

Let $P(x^i)$ and $Q(x^i + dx^i)$ be two neighbouring point in R. The distance ds between P and Q, is given by

$$ds = F(x^i, dx^i).$$

The space equipped with such metric is called a Finsler space.

(ii) Metric tensor g_{ij} is defined by

$$g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2$$

$$= \frac{1}{2} \dot{\partial}_j \dot{\partial}_i F^2$$

$$= g_{ji}.$$

(iii). $$C_{ijk} y^i = \frac{1}{2} \dot{\partial}_k g_{ij} \cdot y^i$$

$$= \frac{1}{4} \dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2 \cdot y^i$$

$$= \dot{\partial}_i \left(\frac{1}{2} g_{jk}\right) y^i$$

$$= 0 \qquad \text{by Euler's theorem.}$$

(iv). $$\gamma^i_{jk} = g^{ip} \cdot \gamma_{jpk}$$
$$= g^{ip} \cdot \frac{1}{2} \left(\partial_j g_{pk} + \partial_k g_{jp} - \partial_p g_{jk} \right)$$

Since the degree of hom. of g_{ij} is zero. Therefore the degree of hom. of γ^i_{jk} is zero.

(v). $$y_i = g_{ij}\, y^j$$
$$= \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2 \cdot y^j$$
$$= \dot{\partial}_j \left(\frac{1}{2} \dot{\partial}_i F^2 \right) \cdot y^j$$
$$= \frac{1}{2} \dot{\partial}_i F^2 \qquad \text{by Euler's theorem.}$$

(vi). $$\frac{\delta X^i}{\delta t} = \frac{d X^i}{dt} + P^i_{jk}\, X^j y^k .$$

(vii). $$G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i$$

and $$G^i = \frac{1}{2} \gamma^i_{jk}\, y^j y^k .$$

Since the degree of hom. of γ^i_{jk} is zero, the degree of hom. of G^i is two. Hence the degree of hom. of G^i_{jk} is zero.

(viii). A connection is called metrical if the covariant derivative of the metric tensor with respect to the connection vanishes identically.

(ix). A Finsler space is called Affinely connected space if
$$G^i_{jkh} = 0 .$$
ie. $$\dot{\partial}_h G^i_{jk} = 0$$
ie. Berwald connection coeff. G^i_{jk} is independent of directional argument.

(x). Since $\mathcal{B}_k g_{jh} = y_r G^r_{jkh}$

and a Finsler space is called a Landsberg space if

$y_r G^r_{jkh} = 0$. Hence $\mathcal{B}_k g_{jh} = 0$.

(xi). Ricci tensor $H_{jk} \overset{\text{def}}{=} H^r_{jkr}$.

(xii). A Finsler space is called symmetric if

$$\mathcal{B}_m H^i_{jkh} = 0.$$

(xiii).
$$\begin{aligned}\pounds y^i &= v^r \mathcal{B}_r y^i + (\dot\partial_r y^i) y^s \mathcal{B}_s v^r - y^r \mathcal{B}_r v^i \\ &= v^r . 0 + \delta^i_r y^s \mathcal{B}_s v^r - y^r \mathcal{B}_r v^i \\ &= 0.\end{aligned}$$

(xiv). An infinitesimal transformation is said to be a conformal transformation if it preserves angle

(xv). Projective deviation tensor W^i_k is defined by

$$W^i_k = H^i_k - H.\delta^i_k - \frac{1}{n+1}(\dot\partial_r H^r_k - \dot\partial_k H).y^i$$

(xvi). A Finsler space F^n $(n>2)$ is projectively flat iff it is a space of scalar curvature.

(B):

Q. 2. Geodesic:

Geodesic is the path of extremum (stationary) distance between two points.

A curve $C: x^i = x^i(t)$ with end points P_0 and P_1 is a Geodesic, if the distance between P_0 and P_1 along C is stationary.

In a Finsler space, the distance betn two points x^i & $x^i + dx^i$ is given by

$$ds = F(x^i, dx^i)$$

$$= F\left(x^i, dt \cdot \frac{dx^i}{dt}\right)$$

$$= \cancel{dt}\ F(x^i, \dot{x}^i)\, dt$$

$$\frac{ds}{dt} = F(x^i, \dot{x}^i)$$

$$\therefore \quad s = \int_{t_0}^{t_1} F(x^i, \dot{x}^i)\, dt \qquad \text{along } C$$

Since C is a Geodesic, this integral is statio

$$\therefore \quad \frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} = 0$$

Now $\dfrac{\partial F}{\partial x^k} = \dfrac{\partial}{\partial x^k}\left[\{g_{ij}(x, \dot{x})\, \dot{x}^i, \dot{x}^j\}^{1/2} \right]$

$$= \frac{1}{2\{g_{ij}(x, \dot{x})\dot{x}^i, \dot{x}^j\}^{1/2}} \left[(\partial_k g_{ij})\, \dot{x}^i \dot{x}^j \right] *$$

$\cancel{g_{ij}\, \delta^i_k}$

$$\frac{\partial F}{\partial \dot{x}^k} = \frac{1}{2\{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j\}^{1/2}}\left\{(\dot{\partial}_k g_{ij})\dot{x}^i\dot{x}^j + g_{ij}\delta^i_k\dot{x}^j + g_{ij}\dot{x}^i\delta^j_k\right\}$$

$$= \frac{1}{2\dot{s}}\left\{2C_{ijk}\dot{x}^i\dot{x}^j + g_{kj}\dot{x}^j + g_{ik}\dot{x}^i\right\}$$

$$= \frac{1}{2\dot{s}}\left\{0 + g_{jk}\dot{x}^j + g_{jk}\dot{x}^j\right\}$$

$$= \frac{1}{\dot{s}} g_{jk}\dot{x}^j$$

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}^k} = -\frac{1}{\dot{s}^2}\ddot{s}\, g_{jk}\dot{x}^j + \frac{1}{\dot{s}}\left(\frac{\partial g_{jk}}{\partial x^h}\frac{\partial x^h}{\partial t} + \frac{\partial g_{jk}}{\partial \dot{x}^h}\frac{\partial \dot{x}^h}{\partial t}\right)\dot{x}^j$$

$$+ \frac{1}{\dot{s}} g_{jk}\frac{d}{dt}\dot{x}^j.$$

$$= -\frac{1}{\dot{s}^2}\ddot{s}\, g_{jk}\dot{x}^j + \frac{1}{\dot{s}}\left\{(\partial_h g_{jk})\dot{x}^h + (\dot{\partial}_h g_{jk})\ddot{x}^h\right\}\dot{x}^j$$

$$+ \frac{1}{\dot{s}} g_{jk}\ddot{x}^j$$

Therefore Euler eqn becomes.

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} = 0$$

$$\frac{1}{2\dot{s}}(\partial_k g_{ij})\dot{x}^i\dot{x}^j + \frac{1}{\dot{s}^2}\ddot{s}\, g_{jk}\dot{x}^j - \frac{1}{\dot{s}}(\partial_h g_{jk}\dot{x}^h + \dot{\partial}_h g_{jk}\ddot{x}^h)\dot{x}^j$$

$$-\frac{1}{\dot{s}} g_{jk}\ddot{x}^j = 0$$

$$\frac{1}{2}(\partial_k g_{ij})\dot{x}^i\dot{x}^j + \frac{1}{\dot{s}}\ddot{s}\, g_{jk}\dot{x}^j - \partial_i g_{jk}\dot{x}^i\dot{x}^j - 2C_{jkh}\ddot{x}^h\dot{x}^j - g_{jk}\ddot{x}^j = 0$$

$$\frac{1}{2}(\partial_k g_{ij})\dot{x}^i\dot{x}^j + \frac{1}{\dot{s}}\ddot{s}\, g_{jk}\dot{x}^j - \partial_i g_{jk}\dot{x}^i\dot{x}^j - 0 - g_{jk}\ddot{x}^j = 0$$

$$\{\text{as } C_{jkh}\dot{x}^j = 0$$

$$\frac{1}{2}\left(\partial_k g_{ij} - 2\partial_i g_{jk}\right)\dot{x}^i\dot{x}^j - g_{jk}\ddot{x}^j + \frac{1}{\dot{s}}\ddot{s}\, g_{jk}\dot{x}^j = 0$$

$$\frac{1}{2}\left(\partial_i g_{jk}\dot{x}^i\dot{x}^j + \partial_i g_{jk}\dot{x}^i\dot{x}^j - \partial_k g_{ij}\dot{x}^i\dot{x}^j\right) + g_{jk}\ddot{x}^j - \frac{1}{\dot{s}}\ddot{s}\, g_{jk}\dot{x}^j = 0$$

$$\frac{1}{2}\left(\partial_i g_{jk}\dot{x}^i\dot{x}^j + \partial_j g_{ik}\dot{x}^j\dot{x}^i - \partial_k g_{ij}\dot{x}^i\dot{x}^j\right) + g_{jk}\ddot{x}^j - \frac{1}{\dot{s}}\ddot{s}\, g_{jk}\dot{x}^j = 0$$

interchanging i & j in second term in Bracket.

$$\Gamma_{ikj}\dot{x}^i\dot{x}^j + g_{jk}\ddot{x}^j - \frac{\ddot{s}}{\dot{s}} g_{jk}\dot{x}^j = 0$$

Multiply by g^{kh} and taking summation over k.

$$\Gamma^h_{ij}\dot{x}^i\dot{x}^j + \delta^h_j\ddot{x}^j - \frac{\ddot{s}}{\dot{s}}\delta^h_j\dot{x}^j = 0$$

or $$\boxed{\ddot{x}^h + \Gamma^h_{ij}\dot{x}^i\dot{x}^j - \frac{\ddot{s}}{\dot{s}}\dot{x}^h = 0}$$

differential eqⁿ of Geodesic.

If we take arc length as parameter

then $\dot{x}^i = x'^i\,\dot{s}$

$$\ddot{x}^i = x''^i\,\dot{s}^2 + x'^i\,\ddot{s}$$

and use $\Gamma^h_{ij}(x^i, \dot{x}^i) = \Gamma^h_{ij}(x^i, x'^i)$

then eqⁿ will become

$$\boxed{x''^i + \Gamma^i_{jk}x'^j x'^k = 0}$$

3. Ricci Commutation Formula for Berwald Covariant differentiation.

$$B_j B_k X^i - B_k B_j X^i = X^r H^i_{rjk} - \dot{\partial}_r X^i \cdot H^r_{jk}$$

where Berwald curvature tensor

$$H^i_{rjk} = \{ \partial_j G^i_{rk} - G^i_{srk} G^s_j + G^i_{sj} G^s_{rk} - j/k \}$$

and Berwald torsion tensor

$$H^r_{jk} = \{ \partial_j G^r_{k} + G^r_{sj} G^s_k - j/k \}$$

Proof.

$$B_j(B_k X^i) = \partial_j (B_k X^i) - (\dot{\partial}_r B_k X^i) G^r_j + B_k X^r G^i_{rj} - B_r X^i G^r_{jk}$$

$$\begin{aligned} B_j B_k X^i = {} & \partial_j \{ \underbrace{\partial_k X^i}_{1} - (\dot{\partial}_r X^i) G^r_k + X^r G^i_{rk} \} \\ & - \dot{\partial}_r \{ \partial_k X^i - (\dot{\partial}_s X^i) G^s_k + X^s G^i_{sk} \} G^r_j \\ & + \{ \partial_k X^r - (\dot{\partial}_s X^r) G^s_k + X^s G^r_{sk} \} G^i_{rj} \\ & - \{ \underbrace{\partial_r X^i}_{1} - \underbrace{(\dot{\partial}_s X^i) G^s_r}_{2} + \underbrace{X^s G^i_{sr}}_{3} \} G^r_{jk} \end{aligned}$$

$$\begin{aligned} B_k B_j X^i = {} & \partial_k \{ \underbrace{\partial_j X^i}_{1} - (\dot{\partial}_r X^i) G^r_j + X^r G^i_{rj} \} \\ & - \dot{\partial}_r \{ \partial_j X^i - (\dot{\partial}_s X^i) G^r_j + X^s G^i_{sj} \} G^r_k \\ & + \{ \partial_j X^r - (\dot{\partial}_s X^r) G^s_j + X^s G^r_{sj} \} G^i_{rk} \\ & - \{ \underbrace{\partial_r X^i}_{1} - \underbrace{(\dot{\partial}_s X^i) G^s_r}_{2} + \underbrace{X^s G^i_{sr}}_{3} \} G^r_{kj} \end{aligned}$$

After some calculation we get the result.

Above result can also be written as

$$X^i_{(j)(k)} - X^i_{(k)(j)} = X^r H^i_{rjk} - (\dot{\partial}_r X^i) H^r_{jk}$$

4. Recurrent Finsler Space.

A Finsler space F_n is said to be reccurent if its curvature tensor satisfies

$$B_m H^i_{jkh} = \lambda_m \cdot H^i_{jkh} \quad ; \quad H^i_{jkh} \neq 0$$

where λ_m is a non-zero covariant vector, called recurrence vector.

<u>Theorem</u>. The recurrence vector λ_m is indep. of the directional argument.

<u>Proof</u>. The recurrence vector λ_m is given by

$$B_m H^i_{jkh} = \lambda_m H^i_{jkh}$$

Contracting i & h, we get

$$B_m H_{jk} = \lambda_m H_{jk}$$

Since y^j is Berwald covariant const. so transvecting above eq^n by y^j, we get

$$B_m H_k = \lambda_m H_k.$$

Diff partially wrt y^j, we get

$$\dot{\partial}_j B_m H_k = \dot{\partial}_j \lambda_m \cdot H_k + \lambda_m \cdot \dot{\partial}_j H_k$$

using comm. formula

$$B_m \dot{\partial}_j H_k - H_r G^r_{jmk} = \dot{\partial}_j \lambda_m \cdot H_k + \lambda_m \cdot \dot{\partial}_j H_k$$

i.e. $B_m H_{jk} - H_r G^r_{jmk} = \dot{\partial}_j \lambda_m \cdot H_k + \lambda_m H_{jk}$

Since $B_m H_{jk} = \lambda_m H_{jk}$

So we get from above eq^n

$$- H_r G^r_{jmk} = \dot\partial_j \lambda_m \cdot H_k$$

Transvecting by y^k and using $G^r_{jmk} y^k = 0$, we get

$$0 = \dot\partial_j \lambda_m \cdot y^k H_k$$

i.e. $0 = \dot\partial_j \lambda_m \cdot (n-1) H$ as $y^k H_k = (n-1)H$

Since $n \neq 1$ & $H \neq 0$

So $\dot\partial_j \lambda_m = 0$

proved.

5. Lie derivative of connection coeff. G^i_{jk} -

$$\bar{x}^i = x^i + \epsilon v^i(x^k) \quad \text{--- (1)}$$

$$\dot{\bar{x}}^i = \dot{x}^i + \epsilon \partial_h v^i \cdot \dot{x}^h \quad \text{--- (2)}$$

Treating the trans. as a general shift

$$\bar{G}^i_{jk} = G^i_{jk}(\bar{x}^h, \dot{\bar{x}}^h)$$

$$= G^i_{jk}(x^h + \epsilon v^h, \dot{x}^h + \epsilon \partial_r v^h \cdot y^r)$$

$$= G^i_{jk}(x^h, \dot{x}^h) + \epsilon v^h \partial_h G^i_{jk} + \epsilon \partial_r v^h \cdot y^r \cdot \dot\partial_h G^i_{jk} + \cdots$$

$$d^v G^i_{jk} = \bar{G}^i_{jk} - G^i_{jk}$$

$$= \epsilon v^h \partial_h G^i_{jk} + \epsilon \partial_r v^h \cdot y^r \dot\partial_h G^i_{jk} + \cdots \quad \text{--- (3)}$$

Treating the trans(1) as a co-ordinate trans.

$$\bar{G}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} G^l_{mn} + \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^k}$$

From (1) $\dfrac{\partial \bar{x}^i}{\partial x^l} = \dfrac{\partial x^i}{\partial x^l} + \epsilon \dfrac{\partial v^i}{\partial x^l} = \delta^i_l + \epsilon \partial_l v^i$

$$\frac{\partial x^m}{\partial \bar{x}^j} = \frac{\partial \bar{x}^m}{\partial \bar{x}^j} - \epsilon \partial_h v^m \cdot \frac{\partial x^h}{\partial \bar{x}^j}$$

$$= \frac{\partial \bar{x}^m}{\partial \bar{x}^j} - \epsilon \partial_h v^m \left(\frac{\partial \bar{x}^h}{\partial \bar{x}^j} - \epsilon \partial_r v^h \frac{\partial x^r}{\partial \bar{x}^j} \right)$$

$$= \delta^m_j - \epsilon \partial_h v^m \left(\delta^h_j - \epsilon \partial_r v^h \frac{\partial x^r}{\partial \bar{x}^j} \right)$$

$$\frac{\partial x^m}{\partial \bar{x}^j} = \delta^m_j - \epsilon \partial_j v^m + \cdots$$

$$\frac{\partial^2 m}{\partial \bar{x}^j \partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k} \left(\delta^m_j - \epsilon \partial_j v^m + \cdots \right)$$

$$= 0 - \epsilon \partial_r \partial_j v^m \frac{\partial x^r}{\partial \bar{x}^k} + \cdots$$

$$= - \epsilon \partial_r \partial_j v^m \left(\delta^r_k - \epsilon \partial_k v^r + \cdots \right) + \cdots$$

$$= - \epsilon \partial_k \partial_j v^m + \cdots$$

So

$$\bar{G}^i_{jk} = \left(\delta^i_l + \epsilon \partial_l v^i \right)\left(\delta^m_j - \epsilon \partial_j v^m + \cdots \right)\left(\delta^n_k - \epsilon \partial_k v^n + \cdots \right) G^l_{mn}$$

$$+ \left(\delta^i_l + \epsilon \partial_l v^i \right)\left(- \epsilon \partial_k \partial_j v^l + \cdots \right)$$

$$= G^i_{jk} + G^l_{jk} \epsilon \partial_l v^i - \epsilon \partial_j v^m G^i_{mk} - \epsilon \partial_k v^n G^i_{jn} - \epsilon \partial_k \partial_j v^i + \cdots$$

$$\therefore d^m G^i_{jk} = \epsilon \partial_l v^i G^l_{jk} - \epsilon \partial_j v^m G^i_{mk} - \epsilon \partial_k v^n G^i_{jn} - \epsilon \partial_k \partial_j v^i + \cdots \tag{4}$$

Hence

$$\pounds G^i_{jk} = \lim_{\epsilon \to 0} \frac{d^v G^i_{jk} - d^m G^i_{jk}}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\left(\epsilon v^h . \partial_h G^i_{jk} + \epsilon \partial_r v^h . y^r \dot{\partial}_h G^i_{jk} + \cdots \right) - \left(\epsilon . \partial_l v^i . G^l_{jk} - \epsilon \partial_j v^m G^i_{mk} - \epsilon \partial_k v^n G^i_{jn} - \epsilon \partial_k \partial_j v^i + \cdots \right)}{\epsilon}$$

$$\pounds G^i_{jk} = v^h \partial_h G^i_{jk} + \partial_r v^h . y^r \dot{\partial}_h G^i_{jk} - \partial_l v^i . G^l_{jk} + \partial_j v^m G^i_{mk}$$
$$+ \partial_k v^n G^i_{jn} + \partial_k \partial_j v^i$$

Since

$$B_k B_j v^i = \partial_k (B_j v^i) - \dot{\partial}_r B_j v^i . G^r_k + B_j v^r . G^i_{rk} - B_r v^i . G^r_{jk}$$

$$B_k B_j v^i = \partial_k \left(\partial_j v^i + v^r G^i_{rj} \right) - \left(B_j \cancel{\dot{\partial}_r v^i}^{\,0} + v^s G^i_{rjs} \right) G^r_k$$
$$+ \left(\partial_j v^r + v^s G^r_{sj} \right) G^i_{rk} - \left(\partial_r v^i + v^s G^i_{sr} \right) G^r_{jk}$$

$$B_k B_j v^i = \partial_k \partial_j v^i + \partial_k v^r \cdot G^i_{rj} + v^r \partial_k G^i_{rj} - v^s G^i_{rjs} G^r_k$$

$$+ \partial_j v^r \cdot G^i_{rk} + v^s G^r_{sj} G^i_{rk} - \partial_r v^i \cdot G^r_{jk} - v^s G^i_{sr} G^r_{jk}$$

$$\therefore \partial_k \partial_j v^i = B_k B_j v^i - \partial_k v^r \cdot G^i_{rj} - v^r \partial_k G^i_{rj} + v^s G^i_{rjs} G^r_k$$

$$- \partial_j v^r \cdot G^i_{rk} - v^s G^r_{sj} G^i_{rk} + \partial_r v^i \cdot G^r_{jk} + v^s G^i_{sr} G^r_{jk}$$

Hence

$$\pounds G^i_{jk} = v^h \partial_h G^i_{jk} + \partial_r v^h \cdot y^r \dot\partial_h G^i_{jk} - \cancel{\partial_r v^i \cdot G^r_{jk}} + \cancel{\partial_j v^m G^i_{mk}}$$

$$+ \cancel{\partial_k v^n \cdot G^i_{jn}} + \left(B_k B_j v^i - \cancel{\partial_k v^r \cdot G^i_{rj}} - v^r \partial_k G^i_{rj} + v^s G^i_{rjs} G^r_k \right.$$

$$\left. - \cancel{\partial_j v^r G^i_{rk}} + v^s G^r_{sj} G^i_{rk} + \cancel{\partial_r v^i \cdot G^r_{jk}} + v^s G^i_{sr} G^r_{jk} \right)$$

$$\pounds G^i_{jk} = B_k B_j v^i + \partial_r v^h \cdot y^r \dot\partial_h G^i_{jk} + v^h \partial_h G^i_{jk} - v^r \partial_k G^i_{rj}$$

$$+ v^s G^i_{rjs} G^r_k - v^s G^r_{sj} G^i_{rk} + v^s G^i_{sr} G^r_{jk}$$

$$= B_k B_j v^i + (B_r v^h - v^s G^h_{sr}) y^r \dot\partial_h G^i_{jk} + v^h \partial_h G^i_{jk}$$

$$- v^r \partial_k G^i_{rj} + v^s G^i_{rjs} G^r_k - v^s G^r_{sj} G^i_{rk} + v^s G^i_{sr} G^r_{jk}$$

$$= B_k B_j v^i + B_r v^h \cdot y^r \dot\partial_h G^i_{jk} + v^r \left(\partial_r G^i_{jk} - \partial_k G^i_{rj} \right.$$

$$\left. + G^i_{rjs} G^s_k - G^i_{hjk} G^h_r + G^i_{sr} G^s_{jk} - G^i_{sk} G^s_{rj} \right\}$$

$$\pounds G^i_{jk} = B_k B_j v^i + v^r H_{jkr} + B_r v^h \cdot y^r G^i_{jkh}$$

Since G^i_{jk} is symmetric in j & k.

So $\boxed{\pounds G^i_{jk} = B_j B_k v^i + H_{kjr} \cdot v^r + B_r v^h \, y^r G^i_{jkh}}$

6. Proof. $\pounds G^i_{jk} = B_j B_k v^i + H^i_{kjr} v^r + G^i_{jkr} y^s B_s v^r$

$$B_h \pounds G^i_{jk} = B_h B_j B_k v^i + B_h H^i_{kjr} v^r + H^i_{kjr} B_h v^r$$
$$+ B_h G^i_{jkr} y^s B_s v^r + G^i_{jkr} y^s B_h B_s v^r$$

interchanging h & j, we have

$$B_j \pounds G^i_{hk} = B_j B_h B_k v^i + B_j H^i_{khr} v^r + H^i_{khr} B_j v^r$$
$$+ B_j G^i_{hkr} y^s B_s v^r + G^i_{hkr} y^s B_j B_s v^r$$

subtracting above two eq^n, we get

$$B_h \pounds G^i_{jk} - B_j \pounds G^i_{hk}$$
$$= \{B_h B_j B_k v^i - B_j B_h B_k v^i\} + v^r (B_h H^i_{kjr} - B_j H^i_{khr})$$
$$+ H^i_{kjr} B_h v^r - H^i_{khr} B_j v^r + (B_h G^i_{jkr} - B_j G^i_{hkr}) y^s B_s v^r$$
$$+ (G^i_{jkr} B_h B_s v^r - G^i_{hkr} B_j B_s v^r) y^s$$

$$= B_k v^r H^i_{rjh} - B_r v^i H^r_{kjh} - (\dot{\partial}_r B_k v^i) H^r_{jh} + v^r (B_h H^i_{kjr} + B_j H^i_{krh})$$
$$+ H^i_{kjr} B_h v^r - H^i_{khr} B_j v^r + (\dot{\partial}_r H^i_{kjh}) y^s B_s v^r$$
$$+ (G^i_{jkr} B_h B_s v^r - G^i_{hkr} B_j B_s v^r) y^s$$

$$= B_k v^r H^i_{rjh} - B_r v^i H^r_{kjh} - (B_k \cancel{\dot{\partial}_r} v^i + v^s G^i_{krs}) H^r_{jh}$$
$$+ v^r (-B_r H^i_{khj} - H^s_{jr} G^i_{ksh} - H^s_{rh} G^i_{ksj} - \cancel{H^s_{hj} G^i_{ksr}}) + H^i_{kjr} B_h v^r$$
$$- H^i_{khr} B_j v^r + \dot{\partial}_r H^i_{kjh} y^s B_s v^r + (\qquad) y^s$$

(using second Bianachi Identity)

Since

$$\pounds H^i_{kjh} = v^r B_r H^i_{kjh} + \dot{\partial}_r H^i_{kjh} \cdot B_s v^r . y^s - H^r_{kjh} B_r v^i + H^i_{rjh} B_k v^r$$

$$+ H^i_{krh} B_j v^r + H^i_{kjr} B_h v^r$$

Hence

$$B_h \pounds G^i_{jk} - B_j \pounds G^i_{hk} = \pounds H^i_{kjh} - H^s_{jr} G^i_{shk} v^r - H^s_{rh} G^i_{ksj} v^r$$

$$+ (G^i_{jkr} B_h B_s v^r - G^i_{hkr} B_j B_s v^r) y^s$$

$$= \pounds H^i_{kjh} + G^i_{jkr} (B_h B_s v^r . y^s + H^r_{hs} v^s)$$

$$- G^i_{hkr} (B_j B_s v^r . y^s + H^r_{js} v^s)$$

$$= \pounds H^i_{kjh} + G^i_{jkr} \pounds G^r_h - G^i_{hkr} \pounds G^r_j$$

Hence

$$B_h \pounds G^i_{jk} - B_j \pounds G^i_{hk} = \pounds H^i_{kjh} + G^i_{jkr} \pounds G^r_h - G^i_{hkr} \pounds G^r_j$$

Since $\pounds G^i_{jk} \Rightarrow 0 \Rightarrow \pounds G^i_j = 0$

Hence by above identity

$$\pounds G^i_{jk} = 0 \quad \Rightarrow \pounds H^i_{jkh} = 0$$

Proved

7. Two processes of Cartan's Covariant differentiation

Let $T_{ij}(x,\dot{x})$ be a second order covariant tensor field. Then according to second postulate

$$DT_{ij} = dT_{ij} - T_{kj}(\Gamma^k_{ih}\,dx^h + C^k_{ih}\,d\dot{x}^h) - T_{ik}(\Gamma^k_{jh}\,dx^h + C^k_{jh}\,d\dot{x}^h) \quad ---(1)$$

Now

$$Dl^i = dl^i + l^k\Gamma^i_{kh}\,dx^h + l^k C^i_{kh}\,d\dot{x}^h$$

$$Dl^i = dl^i + l^k\Gamma^i_{kh}\,dx^h + \cancelto{0}{\frac{\dot{x}^k}{F}C^i_{kh}}\,d\dot{x}^h$$

$$= d\left(\frac{\dot{x}^i}{F}\right) + l^k\Gamma^i_{kh}\,dx^h$$

$$= \frac{1}{F}d\dot{x}^i - \frac{dF}{F^2}\dot{x}^i + l^k\Gamma^i_{kh}\,dx^h$$

$$Dl^i = \frac{1}{F}d\dot{x}^i - \frac{dF}{F}l^i + l^k\Gamma^i_{kh}\,dx^h$$

$$FDl^i = d\dot{x}^i - l^i\,dF + \dot{x}^k\Gamma^i_{kh}\,dx^h$$

$$\therefore\ d\dot{x}^i = FDl^i + \dot{x}^i\frac{dF}{F} - \dot{x}^k\Gamma^i_{kh}\,dx^h \quad ---(2)$$

$\therefore$ From (1), we have

$$DT_{ij} = \partial_h T_{ij}\,dx^h + \dot{\partial}_h T_{ij}\,d\dot{x}^h - T_{kj}(\Gamma^k_{ih}\,dx^h + C^k_{ih}\,d\dot{x}^h)$$
$$- T_{ik}(\Gamma^k_{jh}\,dx^h + C^k_{jh}\,d\dot{x}^h)$$

$$DT_{ij} = \partial_h T_{ij}\,dx^h + \dot{\partial}_h T_{ij}\left(FDl^h + \frac{\dot{x}^h}{F}dF - \dot{x}^k\Gamma^h_{kr}\,dx^r\right)$$
$$- T_{kj}\left\{\Gamma^k_{ih}\,dx^h + C^k_{ih}\left(FDl^h + \cancelto{0}{\frac{\dot{x}^h}{F}}dF - \dot{x}^k\Gamma^h_{kr}\,dx^r\right)\right\}$$
$$- T_{ik}\left\{\Gamma^k_{jh}\,dx^h + C^k_{jh}\left(FDl^h + \cancelto{0}{\frac{\dot{x}^h}{F}}dF - \dot{x}^k\Gamma^h_{kr}\,dx^r\right)\right\}$$

as $C^k_{ih}\,\dot{x}^h = 0$

$$DT_{ij} = \dot\partial_h T_{ij}\, dx^h + \dot\partial_h T_{ij}\left(F D l^h + \frac{dF}{F}\dot x^h - \dot x^p \Gamma^h_{pr}\, dx^r\right)$$
$$- T_{kj}\left(\Gamma^k_{ih}\, dx^h + C^k_{ih} F D l^h - C^k_{ih}\Gamma^h_{pr}\dot x^p dx^r\right)$$
$$- T_{ik}\left(\Gamma^k_{jh}\, dx^h + C^k_{jh} F D l^h - C^k_{jh}\Gamma^h_{pr}\dot x^p dx^r\right)$$

$$DT_{ij} = \left\{F\dot\partial_h T_{ij} - F C^k_{ih} T_{kj} - F C^k_{jh} T_{ik}\right\} D l^h$$
$$+ \left\{\partial_h T_{ij} - \dot\partial_r T_{ij}\cdot \dot x^p \Gamma^r_{ph} - T_{kj}\Gamma^k_{ih} - T_{ik}\Gamma^k_{jh} + C^k_{ir}\Gamma^r_{ph}\dot x^p T_{kj}\right.$$
$$\left. + C^k_{jr}\Gamma^r_{ph}\dot x^p T_{ik}\right\} dx^h + \dot\partial_h T_{ij}\,\dot x^h \frac{dF}{F}$$

$$DT_{ij} = \left\{F\dot\partial_h T_{ij} - A^k_{ih} T_{kj} - A^k_{jh} T_{ik}\right\} D l^h$$
$$+ \left\{\partial_h T_{ij} - \dot\partial_r T_{ij}\cdot \overset{*}{\Gamma}{}^r_{ph}\dot x^p - T_{kj}\overset{*}{\Gamma}{}^k_{ih} - T_{ik}\overset{*}{\Gamma}{}^k_{jh}\right\} dx^h$$
$$+ \dot\partial_h T_{ij}\,\dot x^h \frac{dF}{F}$$

where $F C^k_{ih} = A^k_{ih}$

Let us assume that degree of hom. of T_{ij} is zero.

then $\dot\partial_h T_{ij}\,\dot x^h = 0$.

So $DT_{ij} = T_{ij}|_h\, D l^h + T_{ij|h}\, dx^h$

where $T_{ij}|_h = F\dot\partial_h T_{ij} - A^k_{ih} T_{kj} - A^k_{jh} T_{ik}$ is called Cartan's first covariant derivative of T_{ij} wrt. $\dot x^h$.

& $T_{ij|h} = \partial_h T_{ij} - \dot\partial_r T_{ij}\cdot\overset{*}{\Gamma}{}^r_{ph}\dot x^p - T_{kj}\overset{*}{\Gamma}{}^k_{ih} - T_{ik}\overset{*}{\Gamma}{}^k_{jh}$ is called Cartan's second covariant derivative of T_{ij} wrt. x^h.

$\overset{*}{\Gamma}{}^i_{kh}$ is called Cartan's connection coeff.

Q: Schur's Theorem-

If a Finsler space F_n $(n>2)$ is isotropic at each point of a region, and if the scalar $R(x,\dot{x})$ is indep of its directional arguments $\dot{x}^i$, then the Riemannian curvature is const. throughout that region.

Proof. Bianchi identity for Catan covariant diff. is

$$K^i_{rkh|l} + K^i_{rhl|k} + K^i_{rlk|h}) \cancel{x} + \left(\frac{\partial \Gamma^{*i}_{ks}}{\partial \dot{x}^m} K^m_{rhl} + \frac{\partial \Gamma^{*i}_{hs}}{\partial \dot{x}^m} K^m_{rlk} + \frac{\partial \Gamma^{*i}_{ls}}{\partial \dot{x}^m} K^m_{rkh}\right)\dot{x}^s = 0$$

Since $\dfrac{\partial \Gamma^{*h}_{ij}}{\partial \dot{x}^r}\dot{x}^j = C^h_{ir|j}\dot{x}^j$

$$(K^i_{rkh|l} + K^i_{rhl|k} + K^i_{rlk|h}) + (C^i_{km|s}K^m_{rhl} + C^i_{hm|s}K^m_{rlk} + C^i_{lm|s}K^m_{rkh})\dot{x}^s = 0$$

multiply by $\dot{x}^r$ and noting that $K^i_{rkh}\dot{x}^r = H^i_{kh}$

$$H^i_{kh|l} + H^i_{hl|k} + H^i_{lk|h}) + (C^i_{km|s}H^m_{hl} + C^i_{hm|s}H^m_{lk} + C^i_{lm|s}H^m_{kh})\dot{x}^s = 0$$

Transvecting by $\dot{x}^k$, we get

$$H^i_{h|l} + H^i_{hl|k}\dot{x}^k - H^i_{l|h} + (0 - C^i_{hm|s}H^m_l + C^i_{lm|s}H^m_h)\dot{x}^s = 0$$

Now $C^i_{lm|s}H^m_h - C^i_{hm|s}H^m_l$

$$= C^i_{lm|s}\{F^2R(\delta^m_h - l^m l_h)\} - C^i_{hm|s}\{F^2R(\delta^m_l - l^m l_l)\} = 0$$

So above eq^n becomes

$$H^i_{h|l} + H^i_{hl|k}\dot{x}^k - H^i_{l|h} = 0$$

Contracting i & h,

$$(n-1)H_{|l} + H^i_{il|k}\dot{x}^k - H^i_{l|i} = 0$$

$$(n-1)H_{|l} + \left[\frac{F^2}{3}\{(\dot{\partial}_i R)_{|k}(\delta^i_l - l^i l_l) - (\dot{\partial}_l R)_{|k}(\delta^i_i - l^i l_i)\} + F R_{|k}(l_i\delta^i_l - l_l\delta^i_i)\right]$$

$$- F^2 R_{|i}(\delta^i_l - l^i l_l) = 0$$

$$(n-1)F^2 R_{|l} + \frac{F^2}{3}\{(\dot{\partial}_i R)_{|k}(\delta^i_l - l^i l_l) - (\dot{\partial}_l R)_{|k}(n-1)\}\dot{x}^k + F R_{|k}(1-n)l_l\dot{x}^k$$

$$- F^2 R_{|i}(\delta^i_l - l^i l_l) = 0$$

$$(n-2)F^2 R_{|l} - (n-2)F^2 R_{|k} l^k l_l + \frac{F^2}{3}\{-(n-2)(\dot{\partial}_l R)_{|k} - (\dot{\partial}_i R)_{|k} l^i l_l\}\dot{x}^k = 0$$

$$(n-2)R_{|l} - (n-2)R_{|k} l^k l_l = \frac{1}{3}\left[(n-2)(\dot{\partial}_l R)_{|k} + (\dot{\partial}_i R\cdot\dot{x}^i)_{|k} l_l\cdot\frac{1}{F}\right]\dot{x}^k$$

$$(n-2)R_{|l} - (n-2)R_{|k} l^k l_l = \frac{1}{3}\left[(n-2)(\dot{\partial}_l R)_{|k} + 0\right]\dot{x}^k$$

as degree of hom. of R is zero.

$$R_{|l} - R_{|k} l^k l_l = \frac{1}{3}(\dot{\partial}_l R)_{|k}\cdot\dot{x}^k \qquad ;\text{as } n > 2$$

$$3(R_{|l} - R_{|k} l^k l_l) = (\dot{\partial}_l R)_{|k}\dot{x}^k = 0 \qquad \text{by assump. } \dot{\partial}_l R = 0$$

$$R_{|l} = R_{|k} l^k l_l \qquad \text{----- } (*)$$

diff. wrt. $\dot{x}^h$,

$$0 = R_{|k}\left\{\frac{1}{F}(\delta^k_h - l^k l_h)\cdot l_l + l^k\cdot\frac{1}{F}(g_{hl} - l_h l_l)\right\}$$

$$0 = R_{|h} l_l + R_{|k} l^k g_{lh} - 2R_{|k} l^k l_h l_l$$

$$0 = R_{|k} l^k (g_{lh} - l_h l_l) \qquad \text{using } (*)$$

so either $g_{lh} - l_h l_l = 0$ or $R_{lk} l^k = 0$

if $g_{lh} = l_h l_l$

$$g_{lh} g^{lm} = l_h l_l g^{lm}$$

$$\delta^m_h = l_h l^m$$

Contracting m & h, we get

$$n = 1 \quad \# \text{ as } n > 2$$

so $R_{lk} l^k = 0$

$\Rightarrow R_{lk} l^k l_j = 0$

$\Rightarrow R_{lj} = 0$ using (*)

$$\cancel{\frac{\partial R}{\partial x^l}} \quad \frac{\partial R}{\partial x^j} = 0$$

R is also indep. of position vector. Therefore R is throughout const.

gupta, 12/8/14.